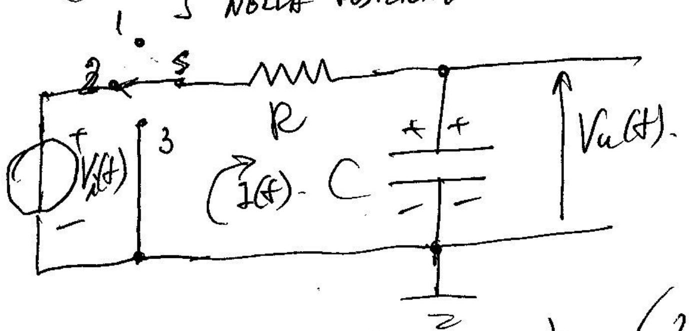
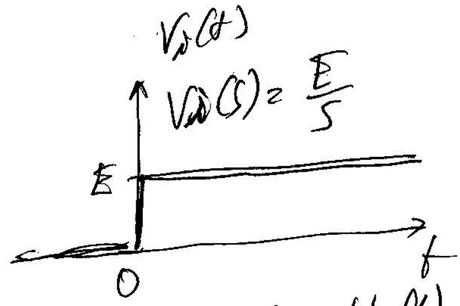
Sistema del I ordine

CIRCUITO RC ECCITATO DA UN GRADINO DI TENSIONE DI AMPIEZZA E

(I METODO)
(equazione differenziale)

1) FASE DI CARICA
(TRANSITORIO DI CARICA)
S NELLA POSIZIONE 2





Equazione del circuito differenziale (2e legge di Kirchhoff)

$$V_i(t) - V_u(t) = R\,I(t)$$

$$C = \frac{Q(t)}{V(t)} \qquad Q(t) = C\,V_u(t)$$

$$I(t) = \frac{dQ(t)}{dt} = C\,\frac{dV_u(t)}{dt}$$

$$V_i(t) - V_u(t) = RC\,\frac{dV_u(t)}{dt}$$

$\mathcal{L}$ - trasformazione.

$$V_i(s) - V_u(s) = RC\,\mathcal{L}\left[\frac{dV_u(t)}{dt}\right]$$

$$V_i(s) - V_u(s) = RC\left[s\,V_u(s) - V_u(0+)\right]$$

$$\frac{E}{s} - V_u(s) = RCs\,V_u(s) - RC\,V_u(0+)$$

$$\frac{E}{s} + RC\,V_u(0+) = V_u(s)\left[1 + RCs\right]$$

$$V_u(s) = \frac{\frac{E}{s} + RC\,V_u(0+)}{1 + RCs} = \frac{E}{s(1+RCs)} + \frac{RC\,V_u(0+)}{1+RCs}$$

$$\frac{E}{s(1+\tau s)} = \frac{A}{s} + \frac{B}{\left(\frac{1}{\tau}+s\right)}$$

$\tau = RC$ poli: $s=0$ e $s=-\frac{1}{\tau}$

$$A = \lim_{s \to 0} \cancel{s} \cdot \frac{E}{\cancel{s}(1+\tau s)} = E$$

$$B = \lim_{s \to -\frac{1}{\tau}} \left(s+\frac{1}{\tau}\right) \frac{E}{s(1+\tau s)} = \lim_{s \to -\frac{1}{\tau}} \cancel{\left(s+\frac{1}{\tau}\right)} \frac{E}{s\tau\cancel{\left(\frac{1}{\tau}+s\right)}} =$$

$$= -\frac{E}{\frac{1}{\tau}\cdot\tau} = -E$$

$$\frac{E}{s(1+\tau s)} = \frac{E}{s} - \frac{E}{s+\frac{1}{\tau}}$$

$$V_u(s) = \frac{E}{s} - \frac{E}{s+\frac{1}{\tau}} + \frac{\tau V_u(0^+)}{1+\tau s} =$$

$$= \frac{E}{s} - \frac{E}{s+\frac{1}{\tau}} + \frac{\cancel{\tau} V_u(0^+)}{\cancel{\tau}\left(s+\frac{1}{\tau}\right)}$$

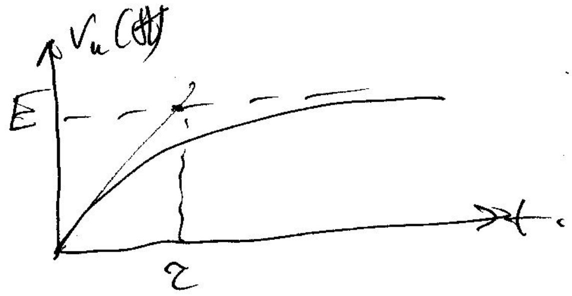
$$V_u(t) = \mathcal{L}^{-1}\frac{E}{s} - \mathcal{L}^{-1}\frac{E}{s+\frac{1}{\tau}} + \mathcal{L}^{-1}\frac{V_u(0^+)}{s+\frac{1}{\tau}} =$$

$$= E - Ee^{-\frac{t}{\tau}} + V_u(0^+)\, e^{-\frac{t}{\tau}} =$$

$$= E\left(1-e^{-\frac{t}{\tau}}\right) + V_u(0^+)\, e^{-\frac{t}{\tau}}$$

se $V_u(0^+) = 0$ $\quad V_u(t) = E\left(1-e^{-\frac{t}{\tau}}\right)$

$W_E = \frac{1}{2} C E^2$ energia elettrostatica accumulata in C per $V_{fin} = E$



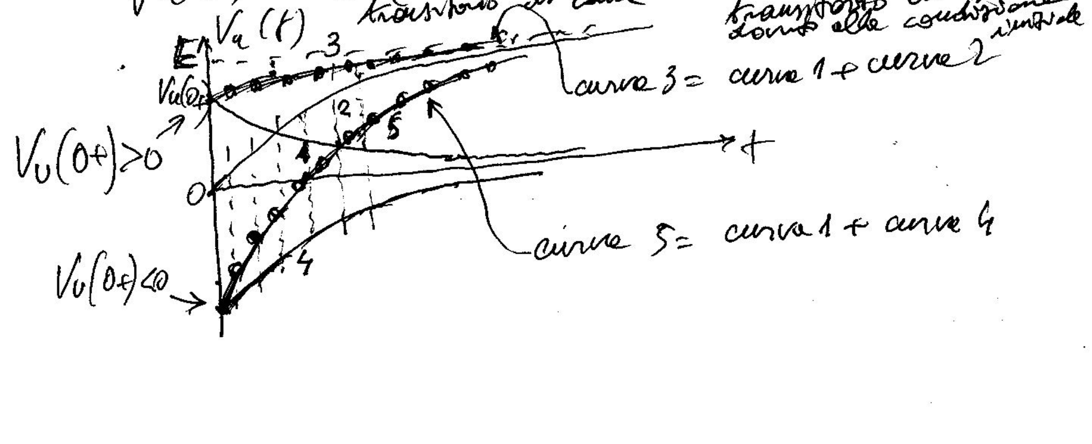
$$I(t) = C\frac{dV_u(t)}{dt} =$$

$$= C\frac{d}{dt}\left[E\left(1-e^{-\frac{t}{\tau}}\right)\right] =$$

$$= C\frac{E}{\tau}e^{-\frac{t}{\tau}} = \frac{CE}{RC} = \frac{E}{R}e^{-\frac{t}{\tau}}$$

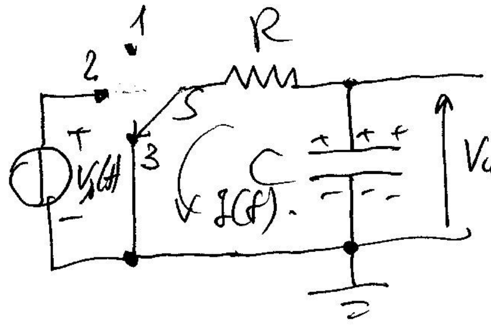
Caso generale: Condensatore inizialmente carico $\rightarrow V_u(0^+) \neq 0$

$$V_u(t) = \underbrace{E\left(1-e^{-\frac{t}{\tau}}\right)}_{\text{transitorio di carica}} + \underbrace{V_u(0^+)\,e^{-\frac{t}{\tau}}}_{\text{transitorio di scarica dovuto alle condizioni iniziali}}$$



2) FASE DI SCARICA

S NELLA POSIZIONE 3.



$V_u(0+) = E$

Eq. differ. del circuito $\quad -V_u(t) = RI(t)$

$I(t) = \frac{dQ(t)}{dt} \Rightarrow I(t) = C\frac{dV_u(t)}{dt}$

$Q(t) = C\,V_u(t)$

$$-V_u(t) = RC\frac{dV_u(t)}{dt}$$

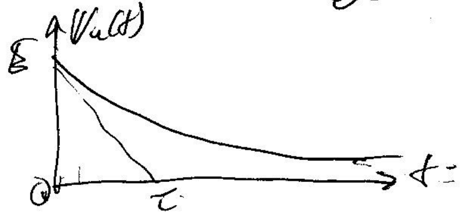
$$-V_u(s) = RC\,\mathcal{L}\left[\frac{dV_u(t)}{dt}\right] = RC\,sV_u(s) - RC\,V_u(0+)$$

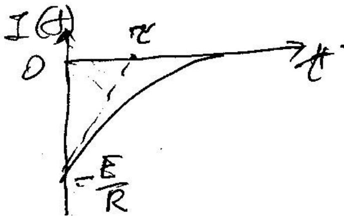
$$RC\,V_u(0+) = (1 + RCs)V_u(s)$$

$$V_u(s) = \frac{RC\,V_u(0+)}{1+RCs} = \frac{\cancel{RC}\,V_u(0+)}{\cancel{RC}\left(\frac{1}{RC}+s\right)} = \frac{V_u(0+)}{s+\frac{1}{\tau}}$$

$$V_u(t) = \mathcal{L}^{-1}V_u(s) = V_u(0+)\,e^{-\frac{t}{\tau}} = E\,e^{-\frac{t}{\tau}}$$

$$I(t) = C\frac{dV_u(t)}{dt} = -\frac{CE}{\tau}e^{-\frac{t}{\tau}} = -\frac{\cancel{C}E}{R\cancel{C}}e^{-\frac{t}{\tau}} = -\frac{E}{R}e^{-\frac{t}{\tau}}$$



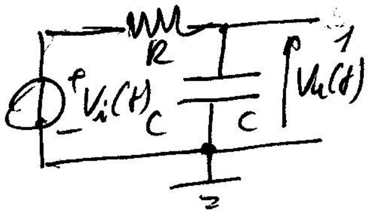


SISTEMI DEL I ORDINE

CIRCUITO RC - RISPOSTA AL GRADINO

II metodo

(equazione integrale)



$$V_i(t) - V_u(t) = R\,I(t)$$

$$I(t) = \frac{dQ(t)}{dt}$$

$$V_u(t) = \frac{Q(t)}{C} = \frac{\int I(t)\,dt}{C} \Rightarrow \frac{\int I(t)\,dt}{C} = \frac{\int_0^t I(t)\,dt}{C} + V_C(0)$$

equazione integrale $\quad V_i(t) - \frac{1}{C}\int I(t)\,dt = R\,I(t)$

$$\mathcal{L}\left[\int I(t)\,dt\right] = \mathcal{L}\left[\int_0^t I(t)\,dt\right] + \mathcal{L}\left[V_C(0)\right]$$

$\mathcal{L}$ - trasformazione

$$V_i(s) - \frac{1}{sC} I(s) - \frac{V_C(0)}{s} = R\,I(s)$$

$$\boxed{\mathcal{L}\left[\int I(t)\,dt\right] = \frac{I(s)}{s} + \frac{V_C(0)}{s}}$$

$$\mathcal{L}[V_i(t)] = V_i(s) = \frac{E}{s}$$

$$\frac{E}{s} - \frac{V_C(0)}{s} = \left(R + \frac{1}{sC}\right) I(s)$$

$$I(s) = \frac{\frac{E}{s} - \frac{V_C(0)}{s}}{R + \frac{1}{sC}} = \frac{E - V_C(0)}{\cancel{s}\cdot\left(\frac{RCs+1}{\cancel{s}C}\right)}$$

$\mathcal{L}^{-1}$ simbolo → antitrasformata di Laplace

$$I(t) = \mathcal{L}^{-1} I(s) = \mathcal{L}^{-1}\left\{\frac{C\,[E - V_C(0)]}{\tau\left(s+\frac{1}{\tau}\right)}\right\} =$$

$\tau = RC$

$$= \mathcal{L}^{-1}\left\{\frac{\cancel{C}}{R\cancel{C}}\,\frac{[E - V_C(0)]}{s+\frac{1}{\tau}}\right\} =$$

$$= \mathcal{L}^{-1}\left[\frac{E - V_C(0)}{R\left(s+\frac{1}{\tau}\right)}\right] = \frac{E - V_C(0)}{R} e^{-\frac{t}{\tau}}$$

$$V_u(t) = V_i(t) - R\,I(t) = E - E e^{-\frac{t}{\tau}} + V_C(0) e^{-\frac{t}{\tau}} =$$

$$= E\left(1 - e^{-\frac{t}{\tau}}\right) + V_C(0)\, e^{-\frac{t}{\tau}}$$

Antitrasformando $I(s)$ si ottiene $I(t)$:

$$I(t) = \mathcal{L}^{-1} I(s) = \mathcal{L}^{-1}\left[\frac{C(E - V_c(0))}{RCs+1}\right] = C\mathcal{L}^{-1}\left[\frac{E - V_c(0)}{\tau\left(s+\frac{1}{\tau}\right)}\right] =$$

$\tau = RC$

$$= \frac{\cancel{C}}{R\cancel{C}}\mathcal{L}^{-1}\left[\frac{E - V_c(0)}{s+\frac{1}{\tau}}\right] = \frac{1}{R}\left[E - V_c(0)\right]e^{-\frac{t}{\tau}}.$$

Sostituendo $I(t)$ nell'equazione del circuito

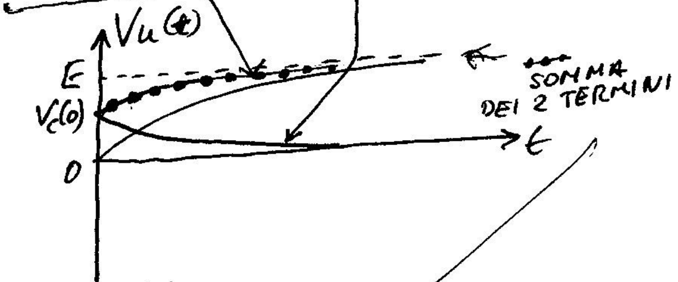
$$V_i(t) - V_u(t) = RI(t) \text{, si ha:}$$

$$V_u(t) = V_i(t) - RI(t) =$$

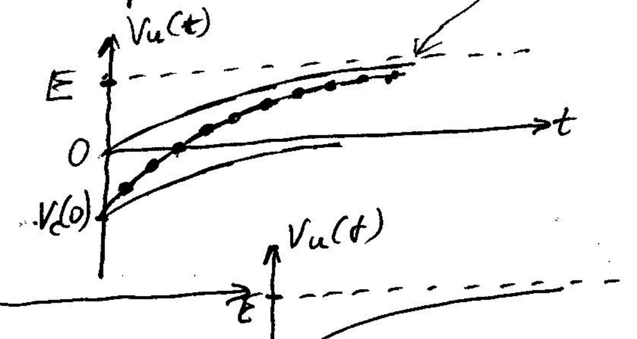
$$= E - \cancel{R}\cdot\frac{1}{\cancel{R}}\left[E - V_c(0)\right]e^{-\frac{t}{\tau}} = E - Ee^{-\frac{t}{\tau}} + V_c(0)e^{-\frac{t}{\tau}};$$

$$V_u(t) = \underbrace{E\left(1 - e^{-\frac{t}{\tau}}\right)} + \underbrace{V_c(0)e^{-\frac{t}{\tau}}}$$ (legge generale di carica del condensatore)

Se $V_c(0) > 0$



Se $V_c(0) < 0$

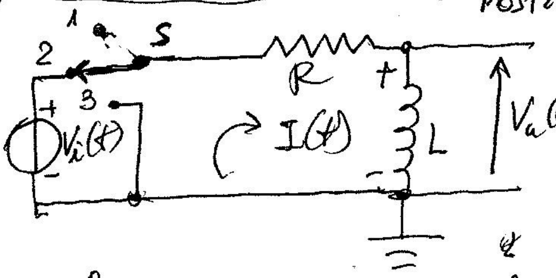


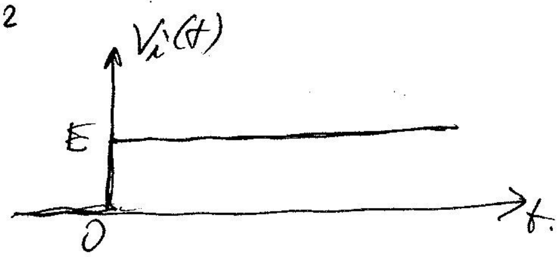
Se $V_c(0) = 0$ si ottiene la nota legge di carica del condensatore

SISTEMA DEL I ORDINE

1) FASE (O TRANSITORIO) DI CHIUSURA — CIRCUITO RL ECCITATO DA UN GRADINO DI AMPIEZZA E

$S \rightarrow$ ~~NELLA~~ POSIZIONE 2





2ª Legge di Kirchhoff.

$$V_i(t) - V_u(t) = R\,I(t)$$ Equazione differenziale del circuito

$$V_i(t) - \frac{d\phi(B)}{dt} = R\,I(t); \qquad \phi(B) = L\,I(t)$$

$$V_i(t) - L\,\frac{dI(t)}{dt} = R\,I(t);$$

$\mathcal{L}$ - trasformazione.

$$\mathcal{L}[V_i(t)] - L\,\mathcal{L}\left[\frac{dI(t)}{dt}\right] = R\,\mathcal{L}[I(t)]$$

$$V_i(s) = \frac{E}{s}$$

$$V_i(s) - L\left[sI(s) - I(0^+)\right] = R\,I(s);$$

$$I(0^+) = 0$$

$$\frac{E}{s} - Ls\,I(s) = R\,I(s);$$

$$\frac{E}{s} = (R + Ls)\,I(s); \qquad \tau = \frac{L}{R}$$

$$I(s) = \frac{E}{s(R+Ls)} = \frac{E}{sL\left(\frac{R}{L}+s\right)} = \frac{E}{sL\left(\frac{1}{\tau}+s\right)}$$

$$I(s) = \frac{E}{sL\left(s+\frac{1}{\tau}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau}}$$

poli: $s = 0$ e $s = -\frac{1}{\tau}$

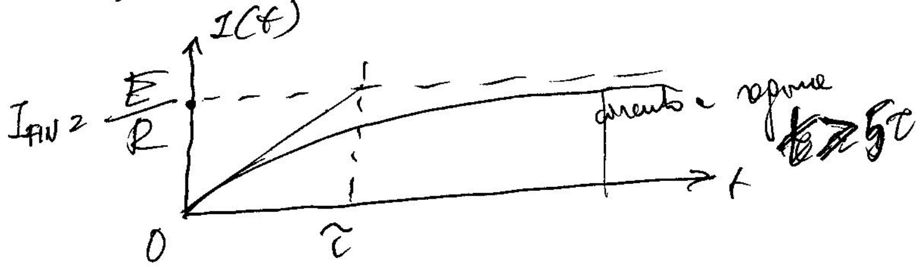
$$A = \lim_{s \to 0} \cancel{s} \frac{E}{\cancel{s}L\left(s+\frac{1}{\tau}\right)} = \frac{E}{L \cdot \frac{1}{\tau}} = \frac{E}{\cancel{L} \cdot \frac{R}{\cancel{L}}} = \frac{E}{R} \quad 2$$

$$B = \lim_{s \to -\frac{1}{\tau}} \cancel{\left(s+\frac{1}{\tau}\right)} \frac{E}{sL\cancel{\left(s+\frac{1}{\tau}\right)}} = \frac{E}{-\frac{1}{\tau} \cdot L} = -\frac{E}{\frac{L}{\frac{L}{R}}} = -\frac{E}{R}$$

$$I(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau}} = \frac{E}{Rs} - \frac{E}{R\left(s+\frac{1}{\tau}\right)}$$

$$I(t) = \mathcal{L}^{-1} I(s) = \mathcal{L}^{-1} \frac{E}{Rs} - \mathcal{L}^{-1} \frac{E}{R\left(s+\frac{1}{\tau}\right)} =$$

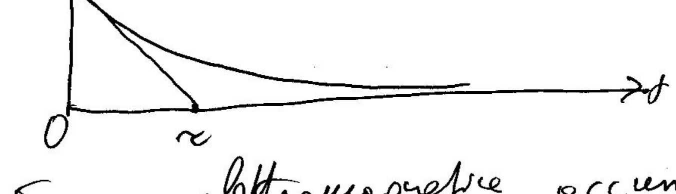
$$= \frac{E}{R} - \frac{E}{R} e^{-\frac{t}{\tau}} = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$



$$V_u(t) = V_i(t) - R\,I(t) = E - R \cdot \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right) =$$

$$= \cancel{E} - \cancel{E} + E e^{-\frac{t}{\tau}}$$

$$V_u(t) = E e^{-\frac{t}{\tau}}$$



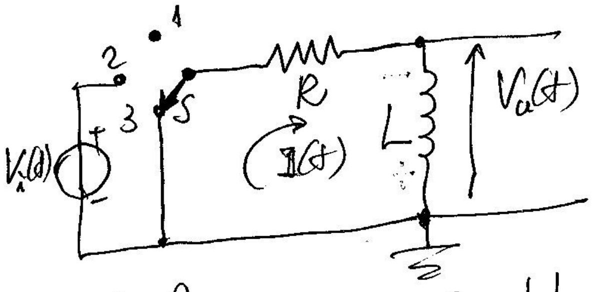
Energia elettromagnetica accumulata in L; $W_{em} = \frac{1}{2} L I_{FIN}^2 = \frac{1}{2} L \left(\frac{E}{R}\right)^2$

con $I_{FIN} = \frac{E}{R}$ (a regime)

3

2) FASE (O TRANSITORIO) DI APERTURA (SOPPRESSIONE DELLA F.E.M.)

S → NELLA POSIZION 3.



2ª legge di Kirchhoff.

$$0 - V_u(t) = R\,I(t). \quad \text{(Equazione diff. del circuito)}$$

$$-L\frac{dI(t)}{dt} = R\,I(t).$$

$\mathcal{L}$ – trasformazione.

$$I(0+) = \frac{E}{R} = I_0$$

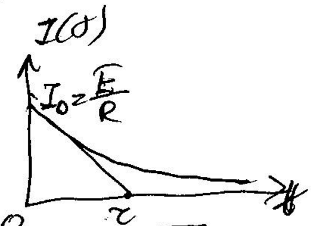
$$-L\,\mathcal{L}\left[\frac{dI(t)}{dt}\right] = R\,\mathcal{L}[I(t)]$$

$$-L\left[s\,I(s) - I(0+)\right] = R\,I(s)$$

$$-Ls\,I(s) + L\,I(0+) = R\,I(s).$$

$$-Ls\,I(s) + L\frac{E}{R} = R\,I(s)$$

$$L\frac{E}{R} = (R + Ls)\,I(s)$$



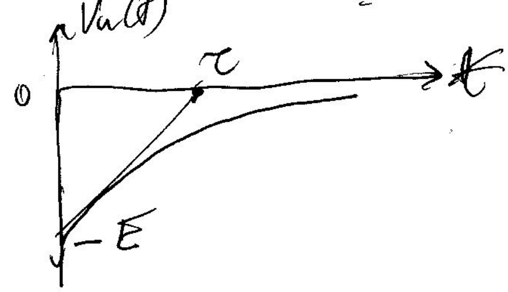
$$I(s) = \frac{L\frac{E}{R}}{R+Ls} = \frac{\cancel{L}\frac{E}{R}}{\cancel{L}\left(\frac{R}{L}+s\right)} = \frac{E}{R\left(\frac{1}{\tau}+s\right)}$$

$$I(t) = \mathcal{L}^{-1}I(s) = \frac{E}{R}\mathcal{L}^{-1}\frac{1}{s+\frac{1}{\tau}} = \frac{E}{R}e^{-\frac{t}{\tau}}$$

4

$$-V_u(t) = R\,I(t)$$

$$V_u(t) = -R\,I(t) = -R\frac{E}{R}e^{-\frac{t}{\tau}} = -E e^{-\frac{t}{\tau}}$$



RISPOSTA DI UN SISTEMA DEL II ORDINE AL GRADINO DI AMPIEZZA a_0 : $V_i(t) = a_0\,\mu(t)$ 1

$$F(s) = \frac{V_u(s)}{V_i(s)} = \frac{\omega_0^2}{s^2+2\gamma\omega_0 s+\omega_0^2} \qquad V_i(s) \longrightarrow \boxed{F(s)} \longrightarrow V_u(s)$$

$$s^2+2\gamma\omega_0 s+\omega_0^2=0$$

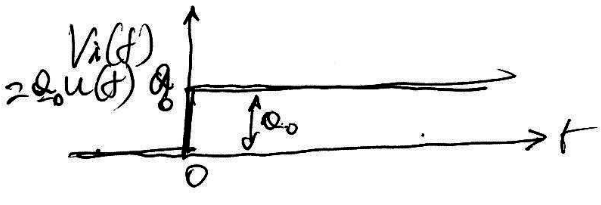
poli: $s = -\omega_0\gamma \pm \sqrt{(\omega_0\gamma)^2-\omega_0^2} = -\omega_0\gamma \pm \omega_0\sqrt{\gamma^2-1}$

γ coefficiente di smorzamento

$\gamma<1$ poli complessi coniugati $p_1 = -\omega_0\gamma + j\omega_0\sqrt{1-\gamma^2}$, $p_2 = -\omega_0\gamma - j\omega_0\sqrt{1-\gamma^2}$

$\gamma=1$ poli reali coincidenti (polo doppio) $p_1 = p_2 = -\omega_0\gamma = -\omega_0$

$\gamma>1$ poli reali e distinti (negativi) $p_1 = -\omega_0\gamma + \omega_0\sqrt{\gamma^2-1}$, $p_2 = -\omega_0\gamma - \omega_0\sqrt{\gamma^2-1}$



$$V_i(s) = \mathcal{L}[a_0\,\mu(t)] = \frac{a_0}{s}$$

Se $\gamma>1$

$$F(s) = \frac{\omega_0^2}{(s-p_1)(s-p_2)}$$

$$V_u(s) = F(s)\,V_i(s) = \frac{a_0\omega_0^2}{s(s-p_1)(s-p_2)} = \frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2}$$

$$A = \lim_{s\to 0} s\,V_u(s) = \frac{\cancel{s}\;a_0\omega_0^2}{\cancel{s}(s-p_1)(s-p_2)} = \frac{a_0\omega_0^2}{p_1p_2}$$

$$B = \lim_{s\to p_1} (s-p_1)V_u(s) = \frac{\cancel{(s-p_1)}\;a_0\omega_0^2}{s\cancel{(s-p_1)}(s-p_2)} = \frac{a_0\omega_0^2}{p_1(p_1-p_2)}$$

$$C = \lim_{s\to p_2} (s-p_2)V_u(s) = \frac{\cancel{(s-p_2)}\;a_0\omega_0^2}{s(s-p_1)\cancel{(s-p_2)}} = \frac{a_0\omega_0^2}{p_2(p_2-p_1)}$$

$$V_u(t) = \mathcal{L}^{-1} V_u(s) = \frac{a_0 \omega_0^2}{p_1 p_2} \mathcal{L}^{-1} \frac{1}{s} + \frac{a_0 \omega_0^2}{p_1(p_1 - p_2)} \mathcal{L}^{-1} \frac{1}{s - p_1} +$$

2

$$+ \frac{a_0 \omega_0^2}{p_2(p_2 - p_1)} \mathcal{L}^{-1} \frac{1}{s - p_2} =$$

$$= \frac{a_0 \omega_0^2}{p_1 p_2} + \frac{a_0 \omega_0^2}{p_1(p_1 - p_2)} e^{p_1 t} + \frac{a_0 \omega_0^2}{p_2(p_2 - p_1)} e^{p_2 t}$$

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se $\gamma = 1$, $p_1 = p_2 = -\omega_0$

$$V_u(s) = \frac{a_0 \omega_0^2}{s(s-p_1)(s-p_2)} = \frac{a_0 \omega_0^2}{s(s+\omega_0)(s+\omega_0)} =$$

$$= \frac{a_0 \omega_0^2}{s(s+\omega_0)^2}$$

$$\frac{a_0 \omega_0^2}{s(s+\omega_0)^2} = \frac{A}{s} + \frac{B}{s+\omega_0} + \frac{C}{(s+\omega_0)^2}$$

si moltiplica a sinistra e destra per $(s+\omega_0)^2$ e si fa il limite per $s \to -\omega_0$

$$\lim_{s \to -\omega_0} \frac{\cancel{(s+\omega_0)^2}\, a_0 \omega_0^2}{s \cancel{(s+\omega_0)^2}} = \lim_{s \to -\omega_0} \frac{(s+\omega_0)^2 A}{s} + \lim_{s \to -\omega_0} \frac{(s+\omega_0)^2 B}{s+\omega_0}$$

$$+ \lim_{s \to -\omega_0} \frac{\cancel{(s+\omega_0)^2} C}{\cancel{(s+\omega_0)^2}}$$

$$- \frac{a_0 \omega_0^{\cancel{2}}}{\cancel{\omega_0}} \rightarrow 0 + 0 + C$$

$$C = - a_0 \omega_0$$

3

Si moltiplica a sinistra e a destra per $(s+\omega_0)$ e si fa il limite per $s \to \infty$

$$\lim_{s\to\infty} \cancel{(s+\omega_0)} \frac{a_0 \omega_0^2}{s(s+\omega_0)^2} = \lim_{s\to\infty} \frac{(s+\omega_0)A}{s} + \lim_{s\to\infty} \frac{(s+\omega_0)B}{(s+\omega_0)} +$$

$$+ \lim_{s\to\infty} \frac{\cancel{(s+\omega_0)}C}{(s+\omega_0)^{\cancel{2}}}$$

$$0 = A + B + 0$$

$$B = -A$$

Si moltiplica a sinistra e a destra per s e si fa il limite per $s \to 0$

$$\lim_{s\to 0} \frac{\cancel{s}\, a_0 \omega_0^2}{\cancel{s}(s+\omega_0)^2} = \lim_{s\to 0} \cancel{s}\frac{A}{\cancel{s}} + \lim_{s\to 0} \frac{sB}{s+\omega_0} + \lim_{s\to 0} \frac{sC}{(s+\omega_0)^2}$$

$$\frac{a_0 \cancel{\omega_0^2}}{\cancel{\omega_0^2}} = A \qquad \frac{sB}{s+\omega_0} \to 0 \qquad \frac{sC}{(s+\omega_0)^2} \to 0$$

$$A = a_0$$

$$B = -A = -a_0$$

$$V_u(s) = \frac{a_0}{s} - \frac{a_0}{(s+\omega_0)} - \frac{a_0\omega_0}{(s+\omega_0)^2}$$

4

$$V_u(t) = \mathcal{L}^{-1} V_u(s) = a_0 - a_0 e^{-\omega_0 t} - a_0\omega_0 e^{-\omega_0 t} t =$$

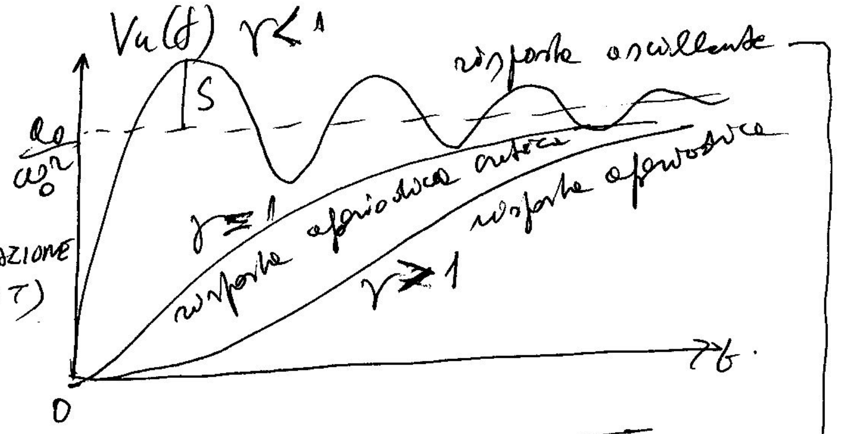
$$= a_0 (1 - e^{-\omega_0 t}) - a_0\omega_0 e^{-\omega_0 t} t$$

risposta aperiodica critica

$$\left[\mathcal{L}\; t\, e^{-\omega_0 t}\right] = F(s+\omega_0) = \frac{1}{(s+\omega_0)^2}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

S: SOVRAELONGAZIONE (OVERSHOOT)



Se $\gamma<1$

$$V_u(t) = a_0 \left\{1 - \frac{e^{-\gamma\omega_0 t}}{\sqrt{1-\gamma^2}} \sin\left[\left(\omega_0\sqrt{1-\gamma^2}\right)t + \varphi\right]\right\}$$

$\varphi = \operatorname{arctg} \frac{\sqrt{1-\gamma^2}}{\gamma}$ (senza dimostrazione) risposta oscillante